Algebraic Geometry - Spring 2026
Sheaves, stalks, and all the love in the world!

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Exercise 1 (Espace étalé of a Presheaf)**.** *Given a presheaf $\mathcal{F}$ on X, define the espace étalé* $\text{Spé}(\mathcal{F})$ *as*

$$\text{Spé}(\mathcal{F}) = \bigcup_{x \in X} \mathcal{F}_x.$$

Define $\pi : \text{Spé}(\mathcal{F}) \to X$ by sending $s \in \mathcal{F}_x$ to x. For each open $U \subseteq X$ and $s \in \mathcal{F}(U)$ define

$$\bar{s} : U \to \text{Spé}(\mathcal{F}), \qquad x \mapsto s_x.$$

Note that $\pi \circ \bar{s} = \text{Id}$, and so, $\bar{s}$ is a section of π over U. Give $\text{Spé}(\mathcal{F})$ *the strongest topology making all maps $\bar{s}$, for all U and all sections $s \in \mathcal{F}(U)$, continuous. Let $\mathcal{F}^+$ be the sheafification of $\mathcal{F}$. Show that for any open $U \subseteq X$, $\mathcal{F}^+(U)$ is the set of continuous sections of* $\text{Spé}(\mathcal{F})$ *over U. Conclude that $\mathcal{F}$ is a sheaf if and only if $\mathcal{F}(U)$ equals the set of such continuous sections.*

Exercise 2 (Support)**.** *Let $\mathcal{F}$ be a sheaf on X and $s \in \mathcal{F}(U)$. Define the* support *of s to be*

$$\text{Supp}(s) = \{x \in U \mid s_x \neq 0\}.$$

Show that $\text{Supp}(s)$ *is closed in U. Define the* support *of $\mathcal{F}$ to be*

$$\text{Supp}(\mathcal{F}) = \{x \in X \mid \mathcal{F}_x \neq 0\}.$$

It need not be closed.

Exercise 3 (Sheaf Hom)**.** *Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves of abelian groups on X. For any open $U \subseteq X$, show that the set*

$$\text{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$$

of homomorphisms of restrictions to U has a natural structure of an abelian group. Show that the presheaf

$$U \mapsto \text{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$$

is a sheaf, called the sheaf Hom, *or* internal Hom, *denoted by $\mathscr{H}om(\mathcal{F}, \mathcal{G})$. What is the stalk of $\mathscr{H}om(\mathcal{F}, \mathcal{G})$ at a point $x \in X$?*

Definition 1.

1. *Let $\mathcal{F}$ be a sheaf of abelian groups on X, and $\mathcal{F}' \subseteq \mathcal{F}$ a subsheaf. The* quotient sheaf $\mathcal{F}/\mathcal{F}'$ *is the sheafification of the presheaf $U \mapsto \mathcal{F}(U)/\mathcal{F}'(U)$.*
2. *For a morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of sheaves of abelian groups, the* kernel sheaf $\text{Ker}(\varphi)$ *is the presheaf $U \mapsto \text{Ker}(\varphi(U))$. It is a sheaf (believe it, or prove it. No, prove it!)*
3. *For a morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of sheaves of abelian groups, define* image sheaf $\text{Im}\,\varphi$ *to be the sheafification of the presheaf $U \mapsto \text{im}(\varphi(U))$.*
4. *We say that a sequence*

$$\mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H}$$

of sheaves of abelian groups is exact if $\text{Im}\,\alpha = \text{Ker}\,\beta$.

Exercise 4 (When in Doubt, Take Stalks). *Let $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ be sheaves of abelian groups on X. Prove the following:*

(a) Let $\mathcal{F}'$ be a subsheaf of $\mathcal{F}$. Then, for any $x \in X$, we have a natural isomorphism

$$(\mathcal{F}/\mathcal{F}')_x \cong \mathcal{F}_x/\mathcal{F}'_x$$

(b) Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism. Then, for any $x \in X$, we have natural isomorphisms:

$$(\operatorname{Ker}\varphi)_x \cong \operatorname{Ker}(\varphi_x), \qquad (\operatorname{Im}\varphi)_x \cong \operatorname{im}(\varphi_x).$$

(c) A sequence

$$\mathcal{F} \to \mathcal{G} \to \mathcal{H}$$

is exact if and only if for every $x \in X$ the induced sequence of stalks

$$\mathcal{F}_x \to \mathcal{G}_x \to \mathcal{H}_x$$

is exact.

Remember this: Stalks see all!

Exercise 5. *(a) Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves such that $\varphi(U)$ is injective for every U. Show that the induced map of associated sheaves $\varphi^+ : \mathcal{F}^+ \to \mathcal{G}^+$ is injective.*

(b) Deduce that if $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves then $\operatorname{Im}\varphi$ can be naturally identified with a subsheaf of $\mathcal{G}$.

Exercise 6. *Give an example of a surjective morphism of sheaves $\varphi : F \to \mathcal{G}$ and an open set U such that*

$$\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$$

is not surjective.

Exercise 7 (Extending a Sheaf by Zero). *Let $Z \subseteq X$ be closed with inclusions*

$$i : Z \hookrightarrow X, \qquad j : U := X \setminus Z \hookrightarrow X.$$

(a) Let $\mathcal{F}$ be a sheaf on Z. Show that

$$(i_*\mathcal{F})_x = \begin{cases} \mathcal{F}_x & x \in Z \\ 0 & x \notin Z \end{cases}$$

We call $i_*\mathcal{F}$ the sheaf obtained by extending $\mathcal{F}$ by zero outside Z.

(b) Let $\mathcal{F}$ be a sheaf on U. Define $j_!\mathcal{F}$ (read it: jay lower shriek) to be the sheaf associated with the presheaf

$$V \mapsto \begin{cases} \mathcal{F}(V) & V \subseteq U \\ 0 & \textit{otherwise} \end{cases}$$

Show the following:

(1)

$$(j_!\mathcal{F})_x = \begin{cases} \mathcal{F}_x & x \in U \\ 0 & x \notin U \end{cases}$$

(2) There is a canonical isomorphism $(j_!\mathcal{F})_{|U} \cong \mathcal{F}$.

(3) $j_!\mathcal{F}$ *is the unique sheaf on* X *with the properties (1) and (2). It is called the the sheaf obtained by* extending $\mathcal{F}$ by zero outside U

(c) Now, let $\mathcal{F}$ *be a sheaf on* X. *Show the that there is an exact sequence*

$$0 \to j_!(\mathcal{F}|_U) \to \mathcal{F} \to i_*(\mathcal{F}|_Z) \to 0.$$

Exercise 8 (Direct Sum)**.** *Let* $\mathcal{F}$ *and* $\mathcal{G}$ *be sheaves of abelian groups on* X. *Show that the presheaf*

$$U \mapsto \mathcal{F}(U) \oplus \mathcal{G}(U)$$

is a sheaf, called the direct sum of $\mathcal{F}$ *and* $\mathcal{G}$ *and denoted by* $\mathcal{F} \oplus \mathcal{G}$. *Show that it is both a product and coproduct in the category of sheaves of abelian groups on* X.

Exercise 9 (Colimit)**.** *Let* $\{\mathcal{F}_i\}$ *be a direct system of sheaves. Define*

$$\varinjlim_i \mathcal{F}_i$$

to be the sheaf associated to the presheaf $U \mapsto \varinjlim_i \mathcal{F}_i(U)$. *Show that it is the colimit of the system* $\{\mathcal{F}_i\}$ *in the category of sheaves on* X.

If X *is a noetherian topological space, show that the presheaf* $U \mapsto \varinjlim_i \mathcal{F}_i(U)$ *is already a sheaf and so there is a canonical isomorphism*

$$\Gamma(X, \varinjlim_i \mathcal{F}_i) \cong \varinjlim_i \Gamma(X, \mathcal{F}_i).$$

Exercise 10 (Inverse Limit)**.** *Let* $\{\mathcal{F}_i\}$ *be an inverse system of sheaves on* X. *Show that the presheaf* $U \mapsto \varprojlim_i \mathcal{F}_i(U)$ *is a sheaf. Show that it is the (inverse) limit of the system* $\{\mathcal{F}_i\}$ *in the category of sheaves on* X.

Exercise 11 (Flasque Sheaves)**.** *A sheaf* $\mathcal{F}$ *on* X *is called* flasque *if for every inclusion of open subsets* $V \subseteq U$, *the restriction map* $\mathcal{F}(U) \to \mathcal{F}(V)$ *is surjective.*

(a) Show that a constant sheaf on an irreducible topological space is flasque.

(b) If

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

is exact and $\mathcal{F}'$ *is flasque, show that for any open* U, *the sequence*

$$0 \to \mathcal{F}'(U) \to \mathcal{F}(U) \to \mathcal{F}''(U) \to 0$$

is exact.

(c) If

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

is exact, and $\mathcal{F}'$ *and* $\mathcal{F}''$ *are flasque, then* $\mathcal{F}$ *is flasque too.*

(d) If $f : X \to Y$ *is continuous and* $\mathcal{F}$ *is a flasque sheaf on* X, *show that* $f_*\mathcal{F}$ *is a flasque sheaf on* Y.

(e) Let $\mathcal{F}$ *be a sheaf on* X. *Define a sheaf* $\mathcal{G}$ *on* X, *called the* sheaf of discontinuous sections, *by sending an open* U *to the set of maps*

$$s : U \to \coprod_{x \in U} \mathcal{F}_x$$

such that $s(x) \in \mathcal{F}_x$. *Show that* $\mathcal{G}$ *is flasque and that there is a natural injection* $\mathcal{F} \hookrightarrow \mathcal{G}$.